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Langevin Picture of Subdiffusion with Infinitely Divisible Waiting Times

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Abstract In this paper we study a Langevin approach to modeling of subdiffusion in the presence of time-dependent external forces. We construct a subordinated Langevin process, whose probability density function solves the subdiffusive fractional Fokker-Planck equation. We generalize the results known for the Lévy-stable waiting times to the case of infinitely divisible waiting-time distributions. Our approach provides a complete mathematical description of subdiffusion with time-dependent forces. Moreover, it allows to study the trajectories of the constructed process both analytically and numerically via Monte-Carlo methodology.

Keywords Subdiffusion · Inverse subordinator · First-passage time · Fractional Fokker-Planck equation · Infinitely divisible distribution

1 Introduction

Recent developments in the area of statistical physics confirm that the classical diffusion models based on Brownian motion fail to provide satisfactory description of many complex systems. Therefore, in the last few years one observes a rapid evolution of various alternative models. Starting with the pioneering papers of Montroll et al., see [18], the physical and mathematical community has shown a growing interest in the development of models for anomalous diffusion processes. The notion of anomalous diffusion refers to a wide family of stochastic processes characterized by certain deviations from the classical Brownian linear time dependence of second moment [16].

Maybe the most relevant subclass of anomalous diffusion processes constitute subdiffusion processes, characterized through the sublinear in time second moment. The list of systems displaying subdiffusive regime is very extensive. Its presence was empirically confirmed in condensed phases, ecology, charge carrier transport in amorphous semiconductors,

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nuclear magnetic resonance, diffusion in percolative and porous systems, transport on fractal geometries and dynamics of a bead in a polymeric network, and protein conformational dynamics, see [16] and references therein.

A common description of subdiffusive transport is in terms of the fractional Fokker-Planck equation (FFPE) derived from the continuous-time random walk (CTRW) [1, 16, 17]. The study of subdiffusive dynamics in the presence of time-dependent force field $F(t)$, giving rise to a modified FFPE, was recently proposed in [22]. The authors of the last paper have derived the following version of the FFPE

$$\frac{\partial w(x,t)}{\partial t} = \left[-F(t)\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2} \right] \Phi w(x,t), \tag{1}$$

$w(x,0) = \delta(x)$, $F(t) \in C([0,\infty))$ and Φ is the appropriate integro-differential operator depending on the waiting-time distribution in the underlying CTRW scenario. Its precise definition will be given in the next section.

Equation (1) is fundamental for statistical physicists in modelling of subdiffusion in time-dependent force fields. It describes the evolution in time of the probability density function (PDF) $w(x,t)$ of some non-Markov stochastic process $Y(t)$. The process $Y(t)$ is called the Langevin process corresponding to FFPE (1), or the stochastic representation of (1).

In this paper we construct rigorously a stochastic process $Y(t)$, whose PDF is a solution of fractional Fokker-Planck equation (1). Our verification method is based on the examination of the moments of the constructed process. The obtained results generalize the ones presented in [12, 13] for Lévy-stable waiting-time distributions. We extend the studies to the general case of infinitely divisible distributions. In Sect. 2 we introduce all the necessary mathematical definitions and properties, which will be helpful in our studies. In Sect. 3 we construct a subordinated Langevin process $Y(t)$ and prove that its PDF solves (1). Since the solution of (1) in the explicit form is not known, in the last section we take advantage of the obtained stochastic representation and introduce a strongly and uniformly convergent approximation scheme. It allows to simulate numerically trajectories of $Y(t)$ and to approximate solutions of (1) using Monte-Carlo methods.

2 Preliminaries

Let us begin with recalling some basic facts concerning infinitely divisible distributions, subordinators and their inverses. We say that a nonnegative random variable T is infinitely divisible, if its Laplace transform takes the form [21]

$$E\left(e^{-uT}\right) = e^{-\Psi(u)},$$

where $\Psi(u)$ is the so-called Lévy exponent. It can be written as

$$\Psi(u) = \lambda u + \int_0^\infty \left(1 - e^{-ux}\right)\nu(dx).$$

Here, $\lambda \geq 0$ is the drift parameter. It is assumed here for simplicity that $\lambda = 0$. The measure $\nu(dx)$ is the appropriate Lévy measure. Some important examples of nonnegative infinitely divisible distributions are: one-sided Lévy stable, Pareto, gamma, Mittag-Leffler, and tempered stable distributions.

Given an infinitely divisible random variable T with the Lévy exponent $\Psi(u)$, we introduce the corresponding stochastic process $\{T_\Psi(t)\}_{t\geq 0}$ via its Laplace transform

$$E\left(e^{-uT_\Psi(t)}\right) = e^{-t\Psi(u)}.$$

T_Ψ is called subordinator. It is a strictly increasing Lévy process. Its increments are nonnegative, independent and stationary. Next, the first-passage time process defined as

$$S_\Psi(t) = \inf\left\{\tau > 0 : T_\Psi(\tau) > t\right\}, \quad t \geq 0, \tag{2}$$

is called the *inverse subordinator*. Since $\lim_{t\to\infty} T_\Psi(t) = \infty$ a.s., S_Ψ is well defined. Inverse subordinators have found various applications in probability theory. Their relationship with local times of some Markov processes is discussed in details in [2]. The connection between inverse subordinators and the theory of renewal processes can be found in [3, 11, 24]. Applications to finance and physics are discussed in [26] and [9, 14, 15, 23], respectively.

Sample paths of $S_\Psi(t)$ are continuous and singular with respect to the Lebesgue measure. Moreover, for every jump of $T_\Psi(\tau)$ there is a corresponding flat period of its inverse. The flat periods of $S_\Psi(t)$ are characteristic for the subdiffusive dynamics, since they represent waiting-times (or the trapping events in which the test particle gets immobilized) in the underlying CTRW scenario. The function

$$U(t) = E(S_\Psi(t))$$

is called the renewal function, [2]. In what follows, we make an additional assumption that there exist a renewal density $u(t)$, i.e. a nonnegative function satisfying $U(t) = \int_0^t u(s)ds$.

FFPE (1) and its solution $w(x,t)$ was constructed in [22] as a limit distribution in the appropriate CTRW scheme. Its derivation was based on the corresponding generalized master equation with two additional physical conditions: the probability conservation in a given state and under transition between different states. The probability balance for the site k reads

$$\dot{p}_k(t) = j_k^+(t) - j_k^-(t).$$

Here the dot denotes the time derivative and $j_k^\pm(t)$ denote the gain and loss currents for a site. The probability conservation for transitions between different sites gives the following relation between the gain current in the state k and loss currents at neighboring sites

$$j_k^+(t) = w_{k-1,k}(t)\, j_{k-1}^-(t) + w_{k+1,k}(t)\, j_{k+1}^-(t).$$

Here, $w_{k-1,k}(t)$ and $w_{k+1,k}(t)$ denote the probabilities of going to the right (from site $k-1$ to site k) and to the left (from site $k+1$ to site k), respectively. Moreover, the considered here physical system is assumed to be infinite and spatially homogeneous.

Given an infinitely divisible distribution $T > 0$ (with the Lévy exponent Ψ) representing the waiting-time distributions in the underlying CTRW scenario, the integro-differential operator Φ in (1) is defined as [22]

$$\Phi f(t) = \frac{d}{dt}\int_0^t M(t-y) f(y)dy \tag{3}$$

for sufficiently smooth function f. Here, the memory kernel $M(t)$ is defined via its Laplace transform

$$\tilde{M}(u) = \int_0^\infty e^{-ut} M(t)dt = \frac{1}{\Psi(u)}. \tag{4}$$

Note that $\frac{1}{\Psi(u)} \sim \frac{e^{-\Psi(u)}}{1-e^{-\Psi(u)}}$ as $u \searrow 0$, therefore the Laplace transform is also written as $\tilde{M}(u) = \frac{e^{-\Psi(u)}}{1-e^{-\Psi(u)}}$, [22]. The case of Lévy-stable waiting-times corresponds to $\Psi(u) \propto u^{\alpha}$, with $\alpha \in (0, 1)$ being the stability parameter. In such case we have

$$\Phi f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-y)^{\alpha-1} f(y) dy.$$

Thus, the operator Φ is equal to the Riemann-Liouville fractional derivative operator ${}_0D_t^{1-\alpha}$, see [20]. This case was discussed in some details in [7, 12, 22] and gave rise to the discovery two significant theoretical properties of subdiffusion, namely: the death of linear response and the field-induced dispersion.

The authors of [22] showed the following result

Proposition 1 [22] *Let $m_n(t) = \int_{-\infty}^{\infty} x^n w(x,t) dx$ denote the moments of the distribution $w(x,t)$ given in (1). Then, $m_n(t)$ satisfy the following recursive formula*

$$m_n(t) = n \int_0^t F(t_1) \Phi m_{n-1}(t_1) dt_1 + \frac{n(n-1)}{2} \int_0^t \Phi m_{n-2}(t_1) dt_1 \tag{5}$$

with $m_0(t) = 1$ and $m_{-1}(t) = 0$.

It turns out that the above formula will play a crucial role in the proof of our main result presented in the next section.

3 Langevin Picture

In this section, we solve the problem of stochastic representation of FFPE (1) with time-dependent force $F(t)$. Recall the definition of $S_\Psi(t)$ in (2). Let us introduce the following subordinated Langevin process

$$Y(t) = X(S_\Psi(t)), \tag{6}$$

where $\{X(\tau)\}_{\tau \geq 0}$ is the solution of the Langevin equation

$$dX(\tau) = F(T_\Psi(\tau)) d\tau + dB(\tau), \quad X(0) = 0. \tag{7}$$

Here, $B(\tau)$ is the standard Brownian motion, independent of the subordinator $T_\Psi(\tau)$ and its inverse $S_\Psi(t)$. We will prove that the PDF of the subordinated Langevin process $Y(t)$ solves FFPE (1). But first, let us discuss the structure of $Y(t) = X(S_\Psi(t))$. The inverse subordinator $S_\Psi(t)$ is related to the operator Φ in FFPE (1), whereas the Langevin process $X(\tau)$ is related to the Fokker-Planck operator (the operator in square brackets in (1)). Processes X and S_Ψ are not independent. We stress the role of the process $T_\Psi(\tau)$, which must appear in (7) in order to fulfill the physical requirement that the time-dependent force $F(t)$ should vary in the real time t. Indeed, since $T_\Psi(\tau)$ is inverse to $S_\Psi(t)$, it cancels the effect of the inverse subordinator on the force F. Therefore, $S_\Psi(t)$ subordinates the process $X(\tau)$ without subordinating the time-dependent force.

The process $Y(t)$ is non-Markov, due to the fact that the inverse subordinator $S_\Psi(t)$ is a local time of some Markov process [2]. Since FFPE (1) describes only one-dimensional distributions, the stochastic representation process (6) gives a complete mathematical picture of subdiffusion. For the case of Markov Lévy driven Langevin systems see [5].

Using the fact that $\{S_\Psi(t) \le \tau\} = \{T_\Psi(\tau) \ge t\}$, and by the standard method of complicating the form of force $F(t)$, one can show that the process $Y(t)$ defined in (6) has the following equivalent representation

$$Y(t) = \int_0^t F(y) dS_\Psi(y) + B(S_\Psi(t)). \tag{8}$$

Here, the integral on the right side of (6) is interpreted pathwise as the Lebesgue-Stieltjes integral. The above formula is very useful from the point of view of the stochastic analysis of trajectories of $Y(t)$ as well as its numerical simulation.

The next theorem is the main result of the paper:

Theorem 1 *Let $B(\tau)$ be the standard Brownian motion and let $S_\Psi(t)$ be the inverse subordinator independent of $B(\tau)$. Then, the PDF of the process*

$$Y(t) = X(S_\Psi(t)),$$

where $X(\tau)$ is defined by the stochastic differential equation (7), *is the solution of the FFPE* (1).

Proof Using (8) we get that the moments of $Y(t)$ can be written as

$$r_n(t) = E(Y^n(t)) = E\left(\left(\int_0^t F(u) dS_\Psi(u) + B(S_\Psi(t))\right)^n\right),$$

$n \in \mathbb{N}$. We prove in the Appendix that $r_n(t)$ satisfy the recursive formula

$$r_n(t) = n \int_0^t F(t_1) \Phi r_{n-1}(t_1) dt_1 + \frac{n(n-1)}{2} \int_0^t \Phi r_{n-2}(t_1) dt_1, \tag{9}$$

with $r_0(t) = 1$ and $r_{-1}(t) = 0$. Then, from (5) we get that moments $m_n(t)$ and $r_n(t)$ coincide. Moreover, for fixed $t_0 > 0$ we have

$$\begin{aligned} r_{2n}(t_0) = E\big(Y^{2n}(t_0)\big) &\le 4^n E\left(\left(\int_0^{t_0} F(u) dS_\Psi(u)\right)^{2n}\right) + 4^n E\left(B^{2n}(S_\Psi(t_0))\right) \\ &\le 4^n M^{2n} E\big(S_\Psi^{2n}(t_0)\big) + 4^n E\big(S_\Psi^n(t_0)\big) E\big(B^{2n}(1)\big) \\ &\le 4^n M^{2n} \frac{e^{t_0}(2n-1)!}{\Psi^{2n}(1)} + 4^n \frac{e^{t_0}(2n)!}{\Psi^n(1) 2^n}, \end{aligned}$$

where $M = \sup_{0 \le s \le t_0} |F(s)|$. Here, we have used the fact that

$$\begin{aligned} E(S_\Psi^n(t_0)) &= \int_0^\infty x^{n-1} \mathbb{P}(S_\Psi(t_0) > x) dx = \int_0^\infty x^{n-1} \mathbb{P}\big(e^{-T_\Psi(x)} > e^{-t_0}\big) dx \\ &\le \int_0^\infty x^{n-1} e^{t_0} e^{-x\Psi(1)} dx \le \frac{e^{t_0} \Gamma(n)}{\Psi^n(1)}. \end{aligned}$$

Consequently, the series $\sum_{n=1}^\infty r_{2n}(t_0) z^n/(2n)!$ is convergent for appropriately small z and the characteristic function of $Y(t)$ is holomorphic in a neighborhood of zero. Therefore, the moments determine the distribution and the PDF of $Y(t_0)$ is equal to the solution $w(x,t_0)$ of FFPE (1). □

4 Simulation

In what follows, we propose a strongly and uniformly convergent approximation scheme of the subordinated Langevin process $Y(t)$ defined in (6). It can be applied to simulate trajectories of $Y(t)$ and to approximate solutions of (1) using Monte-Carlo methods.

Let $\delta > 0$ be the step length. We define the following approximation $S_{\Psi,\delta}(t)$ of the inverse subordinator $S_\Psi(t)$

$$S_{\Psi,\delta}(t) = (\min\{n \in \mathbb{N} : T_\Psi(\delta n) > t\} - 1)\,\delta. \tag{10}$$

Next, we introduce the following approximation of the process $Y(t)$

$$Y_\delta(t) = \int_0^t F(u)dS_{\Psi,\delta}(u) + B(S_{\Psi,\delta}(t)), \quad t \geq 0. \tag{11}$$

The next result shows the uniform convergence and verifies the order of convergence of the above approximations. At this point, we make an additional assumption that the force F is of bounded variation on every interval $[0,t]$ and continuous. The function

$$V_F(t) = \sup_P \sum_{i=1}^n |F(t_i) - F(t_{i-1})|,$$

where $P = \{$all partitions of the interval $[0,t]\}$, is called the total variation of F.

Theorem 2 *For every $T > 0$ the introduced approximation processes $S_{\Psi,\delta}(t)$ and $Y_\delta(t)$ satisfy the following conditions*

(i)

$$\sup_{0\leq s\leq T} |S_\Psi(s) - S_{\Psi,\delta}(s)| \leq \delta \quad a.s.$$

(ii) *Let $0 < q < 1/2$. Then, for appropriately small $\delta > 0$*

$$\sup_{0\leq s\leq T} |Y(s) - Y_\delta(s)| \leq C\delta + \delta^q \quad a.s.,$$

where $C = \sup_{0\leq s\leq T} |F(s)| + 2V_F(T) - F(T) + F(0)$.

(iii)

$$E(|Y(T) - Y_\delta(T)|) \leq C_1\delta + C_2\delta^{1/2},$$

where $C_1 = |F(T)| + 2V_F(T) - F(T) + F(0)$ and $C_2 = E(|B(1)|)$.

The proof follows the one presented in [12] for the Lévy-stable case.

To evaluate numerically the process $S_{\Psi,\delta}(t)$, one must simulate the values $T_\Psi(\delta n)$, $n = 1, 2, \ldots$. Since T_Ψ is a Lévy process, this can be done by one of the methods presented in [19]. Some more efficient methods can be used for particular distributions (for the case of stable distributions see [8, 25], for the geometric stable laws see [10]).

To evaluate the process $Y_\delta(t)$ it is enough to note that $S_{\Psi,\delta}(t)$ is a scaled renewal process. Thus, the integral in (11) can be written as

$$\int_0^t F(u)dS_{\Psi,\delta}(u) = \delta \sum_{n=1}^{N} F(T_\Psi(\delta n)).$$

Here, N is an integer number such that $T_\Psi(\delta N) < t \leq T_\Psi(\delta(N+1))$. Now, the sum on the right side of the above formula as well as the trajectories of Brownian motion $B(\tau)$, can be easily calculated numerically. Therefore, sample paths of $Y_\delta(t)$ are simulated very efficiently.

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Appendix

Proof of formula (9) For the renewal function $U(t) = E(S_\Psi(t))$ we have

$$U(t) = \int_0^\infty \mathbb{P}(T_\Psi(x) < t)dx = \int_0^\infty \int_0^t f_{T_\Psi(x)}(y)dydx,$$

where $f_{T_\Psi(x)}$ is the PDF of the random variable $T_\Psi(x)$. Therefore, the Laplace transform of the renewal density $u(t)$ equals

$$\tilde{u}(s) = \int_0^\infty \int_0^\infty e^{-st} f_{T_\Psi(x)}(t)dtdx = \int_0^\infty e^{-x\Psi(s)} = \frac{1}{\Psi(s)}.$$

Therefore, the operator Φ in (1) has the form

$$\Phi f(t) = \frac{d}{dt}\int_0^t u(t-y)f(y)dy. \tag{12}$$

The above formula gives us the useful link between the operator Φ and the inverse subordinator $S_\Psi(t)$.

Let us now put

$$a_n(t) = E\left(\left(\int_0^t F(t_1)dS_\Psi(t_1)\right)^n\right),$$

with $n \in \mathbb{N}$. We will first show that

$$a_n(t) = n\int_0^t F(t_1)\Phi a_{n-1}(t_1)dt_1. \tag{13}$$

By iterating the change of variable formula for Lebesque-Stieltjes integral, we get

$$\left(\int_0^t F(t_1)dS_\Psi(t_1)\right)^n = n! \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} F(t_1)\dots F(t_n)dS_\Psi(t_n)\dots dS_\Psi(t_1). \tag{14}$$

Now, we introduce the random measure on $[0,\infty)$ by putting $\Pi((s,t]) = S_\Psi(t) - S_\Psi(s)$, $t > s \geq 0$. Next, let $\{C(t)\}_{t\geq 0}$ be the Cox process directed by Π. Thus, conditionally on

$\Pi = \lambda$, $C(t)$ is equal in distribution to the inhomogeneous Poisson process with intensity λ. In such setting $C(t)$ is the renewal process with the renewal function [6]

$$E(C(t)) = E(S_\Psi(t)) = U(t).$$

Moreover, for the renewal process $C(t)$ the following property holds [4]

$$E(dC(t_1)\dots dC(t_n)) = \prod_{i=1}^{n} U(dt_i - t_{i+1}),$$

where $t_1 > t_2 > \dots > t_n > t_{n+1} = 0$. By the fact that the factorial moments of every Cox process are equal to the ordinary moments of its directing measure, see [4], we get

$$E(dS_\Psi(t_1)\dots dS_\Psi(t_n)) = \prod_{i=1}^{n} U(dt_i - t_{i+1}).$$

Thus, using (14) with the above result, we have

$$\begin{aligned} a_n(t) &= E\left(n! \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} F(t_1)\dots F(t_n) dS_\Psi(t_n)\dots dS_\Psi(t_1)\right) \\ &= n! \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \prod_{i=1}^{n} F(t_i) u(t_i - t_{i+1}) dt_n \dots dt_1. \end{aligned}$$

Consequently,

$$\begin{aligned} a_n(t) &= n \int_0^t F(t_1) \int_0^{t_1} u(t_1 - t_2) \frac{d}{dt_2} a_{n-1}(t_2) dt_2 dt_1 \\ &= n \int_0^t F(t_1) \Phi a_{n-1}(t_1) dt_1. \end{aligned}$$

Next, let us put

$$b_n(t) = E(B^n(S_\Psi(t))).$$

Clearly, for $n = 2m - 1$, $m \in \mathbb{N}$, we have $b_n(t) = 0$. For $n = 2m$, by conditioning arguments in combination with (13), we obtain

$$\begin{aligned} b_n(t) &= E(S_\Psi^m(t)) E(B^n(1)) = m(n-1) \int_0^t \Phi E(S_\Psi^{m-1}(t_1)) E(B^{n-2}(1)) dt_1 \\ &= \frac{n(n-1)}{2} \int_0^t \Phi b_{n-2}(t_1) dt_1. \end{aligned} \tag{15}$$

Let us now define

$$c_{k,n}(t) = E\left((B(S_\Psi(t)))^k \left(\int_0^t F(t_1) dS_\Psi(t_1)\right)^n\right),$$

with $k, n \in \mathbb{N}$. If $k = 2m - 1$, $m \in \mathbb{N}$, then by conditioning $c_{k,n}(t) = 0$. For $k = 2m$, $m \in \mathbb{N}$, we will show that

$$c_{k,n}(t) = n \int_0^t F(t_1) \Phi c_{k,n-1}(t_1) dt_1 + \frac{k(k-1)}{2} \int_0^t \Phi c_{k-2,n}(t_1) dt_1. \tag{16}$$

We have

$$c_{k,n}(t) = E(B^k(1))E\left(S_\Psi^{k/2}(t)\left(\int_0^t F(t_1)dS_\Psi(t_1)\right)^n\right).$$

Integrating by parts and iterating the change of variables formula, we get

$$\begin{aligned} &S_\Psi^{k/2}(t)\left(\int_0^t F(t_1)dS_\Psi(t_1)\right)^n \\ &\quad = n!\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\prod_{i=1}^n F(t_i)S_\Psi^{k/2}(t_1)dS_\Psi(t_n)\dots dS_\Psi(t_1) \\ &\qquad + n!k/2\int_0^t\int_0^{t_1}\cdots\int_0^{t_n}\prod_{i=2}^{n+1} F(t_i)S_\Psi^{k/2-1}(t_1)dS_\Psi(t_{n+1})\dots dS_\Psi(t_1). \end{aligned}$$

Moreover, for every $q \in \mathbb{N}$, we have

$$S_\Psi^q(t_1) = q!\int_0^{t_1}\cdots\int_0^{t_q} dS_\Psi(t_{q+1})\dots dS_\Psi(t_2).$$

Now, repeating the arguments from the case $a_n(t)$ in (13), we get

$$\begin{aligned} c_{k,n}(t) &= n\int_0^t F(t_1)\int_0^{t_1} u(t_1-t_2)\frac{d}{dt_2}c_{k,n-1}(t_2)dt_2dt_1 \\ &\quad + \frac{k(k-1)}{2}\int_0^t\int_0^{t_1} u(t_1-t_2)\frac{d}{dt_2}c_{k-2,n}(t_2)dt_2dt_1 \\ &= n\int_0^t F(t_1)\Phi c_{k,n-1}(t_1)dt_1 + \frac{k(k-1)}{2}\int_0^t \Phi c_{k-2,n}(t_1)dt_1, \end{aligned}$$

which yields (16).

Finally, using the Newton's binomial expansion, we get that formula (9) is equivalent to

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} c_{k,n-k}(t) &= n\sum_{k=0}^{n-1}\binom{n-1}{k}\int_0^t F(t_1)\Phi c_{k,n-k-1}(t_1)dt_1 \\ &\quad + \frac{n(n-1)}{2}\sum_{k=0}^{n-2}\binom{n-2}{k}\int_0^t \Phi c_{k,n-k-2}(t_1)dt_1. \end{aligned} \tag{17}$$

Thus, taking advantage of (13), (15) and (16), we get that (17) and equivalently (9) hold. □

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